e content for students of patliputra university

B. Sc. (Honrs) Part 1paper 2

Subject:Mathematics

Title/Heading of topic:Tests for convergence of infinite series (Ratio test, Root test)

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3. D' Alembert's Ratio Test

Let $\sum_{n=1}^{\infty} u_n$ be a positive term series such that

$$\lim_{n\to\infty} \frac{u_{n+1}}{u_n} = l$$

Then (*i*) $\sum_{n=1}^{\infty} u_n$ converges if $l < 1$

(*ii*) $\sum_{n=1}^{\infty} u_n$ diverges if $l > 1$

(*iii*) Test fails if $l = 1$

Example 8 Test the convergence of the following series:

$(i)\frac{1}{3}+\frac{1}{2.3^2}+\frac{1}{3.3^3}+\frac{1}{4.3^4}.....$ $(ii)\ \frac{1^2 2^2}{1}+\frac{2^2 3^2}{2!}+\frac{3^2 4^2}{3!}+\frac{4^2 5^2}{4!}.....$ $(iii)\sum_{n=1}^{\infty}\frac{n!}{n^n}$

Solution: (i)Here $u_n=\frac{1}{n3^n}\ \Rightarrow u_{n+1}=\frac{1}{(n+1)3^{n+1}}$

$$\text{Then } \lim_{n\to\infty}\frac{u_{n+1}}{u_n}=\lim_{n\to\infty}\frac{n3^n}{(n+1)3^{n+1}}=\lim_{n\to\infty}\frac{n}{(n+1)3}$$

$$=\lim_{n\to\infty}\frac{1}{3\left(1+\frac{1}{n}\right)}=0<1$$

Hence by Ratio test ,the given series converges.

(ii) Here $u_n=\frac{n^2(n+1)^2}{n!}\Rightarrow u_{n+1}=\frac{(n+1)^2(n+2)^2}{(n+1)!}$

$$\text{Then } \lim_{n\to\infty}\frac{u_{n+1}}{u_n}=\lim_{n\to\infty}\frac{(n+1)^2(n+2)^2}{(n+1)!}\cdot\frac{n!}{n^2(n+1)^2}$$

$$=\lim_{n\to\infty}\frac{(n+2)^2}{(n+1)}\cdot\frac{1}{n^2}=\lim_{n\to\infty}\frac{1}{(n+1)}\cdot\left(\frac{1+\frac{2}{n}}{1}\right)^2=0<1$$

Hence by Ratio test , the given series converges.

(iii) Here $u_n=\frac{n!}{n^n}\ \Rightarrow u_{n+1}=\frac{(n+1)!}{(n+1)^{(n+1)}}$

$$\text{Then } \lim_{n\to\infty}\frac{u_{n+1}}{u_n}=\lim_{n\to\infty}\frac{(n+1)!}{(n+1)^{(n+1)}}\cdot\frac{n^n}{n!}=\lim_{n\to\infty}\frac{n^n}{(n+1)^n}$$

$$=\lim_{n\to\infty}\frac{1}{\left(1+\frac{1}{n}\right)^n}=\frac{1}{e}=\frac{1}{2.718}<1$$

Hence by Ratio test , the given series converges.

Example 9 Test the convergence of the following series:

$(i)\ \frac{1}{7}+\frac{2!}{7^2}+\frac{3!}{7^3}+\frac{4!}{7^4}.....$ $(ii)\ \left(\frac{1}{3}\right)^2+\left(\frac{1.2}{3.5}\right)^2+\left(\frac{1.2.3}{3.5.7}\right)^2+\left(\frac{1.2.3.4}{3.5.7.9}\right)^2+.....$

Solution: (i) Here $u_n=\frac{n!}{7^n}\Rightarrow u_{n+1}=\frac{(n+1)!}{7^{n+1}}$

$$\text{Then } \lim_{n\to\infty}\frac{u_{n+1}}{u_n}=\lim_{n\to\infty}\frac{(n+1)!}{7^{(n+1)}}\cdot\frac{7^n}{n!}$$

$$=\lim_{n\to\infty}\frac{n+1}{7}=\infty>1$$

Hence by Ratio test , the given series diverges.

(ii) Here $u_n=\left[\frac{1.2.3.4.....n}{3.5.7.9....(2n+1)}\right]^2 \Rightarrow u_{n+1}=\left[\frac{1.2.3.4.....n(n+1)}{3.5.7.9....(2n+1)(2n+3)}\right]^2$

$$\text{Then } \lim_{n\to\infty}\frac{u_{n+1}}{u_n}=\lim_{n\to\infty}\left(\frac{n+1}{2n+3}\right)^2=\frac{1}{2^2}=\frac{1}{4}<1$$

Hence by Ratio test , the given series converges.

Example 10 Test the convergence of the following series:

(i) $\frac{x}{\sqrt{5}}+\frac{x^3}{\sqrt{7}}+\frac{x^5}{\sqrt{9}}+\frac{x^7}{\sqrt{11}}+.....$ (ii) $\frac{x}{1.3}+\frac{x^2}{2.4}+\frac{x^3}{3.5}+\frac{x^4}{4.6}+\cdots(x>0)$

Solution: (i) Here $u_n=\frac{x^{2n-1}}{\sqrt{2n+3}} \Rightarrow u_{n+1}=\frac{x^{2n+1}}{\sqrt{2n+5}}$

$$\text{Then } \lim_{n\to\infty}\frac{u_{n+1}}{u_n}=\lim_{n\to\infty}\frac{x^{2n+1}}{\sqrt{2n+5}}\frac{\sqrt{2n+3}}{x^{2n-1}}=x^2$$

Hence by Ratio test , the given series converges if $x^2<1$ and diverges if $x^2>1$.

Test fails if $x^2=1$. i.e. $x=1$

When $x=1$, $u_n=\frac{1}{\sqrt{2n+3}}$

Let $v_n=\frac{1}{\sqrt{n}}$. Now consider $\frac{u_n}{v_n}=\frac{\sqrt{n}}{\sqrt{2n+3}}$

$$\Rightarrow \lim_{n\to\infty}\frac{u_n}{v_n}=\lim_{n\to\infty}\frac{\sqrt{n}}{\sqrt{2n+3}}$$

$$=\frac{1}{2}\text{(which is a finite and non zero number)}$$

Since $\sum_{n=1}^{\infty}v_n=\sum_{n=1}^{\infty}\frac{1}{\sqrt{n}}$ diverges (as $p=\frac{1}{2}<1$) $\therefore \sum_{n=1}^{\infty}u_n$ also diverges for $x=1$ (by Limit form test).

$\therefore$ the given series converges for $x < 1$ and diverges for $x \geq 1$.

(ii) Here $u_n = \frac{x^n}{n(n+2)} \Rightarrow u_{n+1} = \frac{x^{n+1}}{(n+1)(n+3)}$

Then $\lim_{n\to\infty} \frac{u_{n+1}}{u_n} = \lim_{n\to\infty} \frac{n(n+2)}{(n+1)(n+3)}x = x$

Hence by Ratio test , the given series converges if $x < 1$ and diverges if $x > 1$

Test fails if $x = 1$

When $x = 1$, $u_n = \frac{1}{n(n+2)}$

Let $v_n = \frac{1}{n^2}$.

$\Rightarrow \lim_{n\to\infty} \frac{u_n}{v_n} = \lim_{n\to\infty} \frac{n^2}{n(n+2)}$

$= 1$ (which is a finite and non zero number)

Since $\sum_{n=1}^{\infty} v_n = \sum_{n=1}^{\infty} \frac{1}{n^2}$ converges (as $p = 2 > 1$)

$\therefore \sum_{n=1}^{\infty} u_n$ also converges for $x=1$ (by Limit form test).

$\therefore$ the given series converges for $x \leq 1$ and diverges for $x > 1$.

3. Cauchy's n th Root Test

Let $\sum_{n=1}^{\infty} u_n$ be a positive term series such that

$$\lim_{n\to\infty} (u_n)^{1/n} = l$$

Then (i) $\sum_{n=1}^{\infty} u_n$ converges if $l < 1$

(ii) $\sum_{n=1}^{\infty} u_n$ diverges if $l > 1$

(iii) Test fails if $l = 1$

Example 11 Test the convergence of the following series:

(*i*) $1+\frac{1}{2^2}+\frac{1}{3^3}+\frac{1}{4^4}\ldots..$ (*ii*) $\sum_{n=1}^{\infty}\left(\frac{n}{n+1}\right)^{n^2}$ (*iii*) $\sum_{n=1}^{\infty}5^{-n-(-1)^n}$

Solution: (*i*) Here $u_n=\frac{1}{n^n}$

$$\Rightarrow \lim_{n\to\infty}(u_n)^{1/n}=\lim_{n\to\infty}\frac{1}{n}=0<1$$

Hence by Cauchy's root test, the given series converges.

(*ii*) Here $u_n=\left(\frac{n}{n+1}\right)^{n^2}\Rightarrow \lim_{n\to\infty}(u_n)^{1/n}=\lim_{n\to\infty}\left(\frac{n}{n+1}\right)^n$

$$=\lim_{n\to\infty}\left(\frac{1}{1+\frac{1}{n}}\right)^n=\frac{1}{e}<1$$

Hence by Cauchy's root test, the given series converges.

(*iii*) Here $u_n=5^{-n-(-1)^n}\Rightarrow \lim_{n\to\infty}(u_n)^{1/n}=\lim_{n\to\infty}5^{-\{n+(-1)^n\}.1/n}$

$$=\lim_{n\to\infty}5^{-\left\{1+\frac{(-1)^n}{n}\right\}}=5^{-1}$$

$$=\frac{1}{5}<1$$

Hence by Cauchy's root test, the given series converges.

Example 12 Test the convergence of the following series:

$$\left(\frac{2^2}{1^2}-\frac{2}{1}\right)^{-1}+\left(\frac{3^3}{2^3}-\frac{3}{2}\right)^{-2}+\left(\frac{4^4}{3^4}-\frac{4}{3}\right)^{-3}+\cdots$$

Solution: Here $u_n=\left[\left(\frac{n+1}{n}\right)^{n+1}-\frac{n+1}{n}\right]^{-n}$

$$\Rightarrow \lim_{n\to\infty}(u_n)^{1/n}=\lim_{n\to\infty}\left[\left(\frac{n+1}{n}\right)^{n+1}-\frac{n+1}{n}\right]^{-1}$$

$$= \lim_{n\to\infty} \left(\frac{n+1}{n}\right)^{-1} \left[\left(\frac{n+1}{n}\right)^{n} - 1\right]^{-1}$$

$$= \lim_{n\to\infty} \left(1 + \frac{1}{n}\right)^{-1} \left[\left(1 + \frac{1}{n}\right)^{n} - 1\right]^{-1}$$

$$= \frac{1}{e-1} < 1$$

Hence by Cauchy's root test, the given series converges.